# Symmetries and invariant solutions of the planar paraxial wave equation in photosensitive media

M. Senthilvelan,[1,2] L. Poladian,[2] and C. Martijn de Sterke[1]
[1]*School of Physics, The University of Sydney, NSW 2006, Australia*
[2]*Optical Fibre Technology Centre, The University of Sydney, Australian Technology Park, NSW 1430, Australia*
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We study the equations describing planar self-written waveguides through group theoretical methods. We show the equations are nonintegrable through Painlevé analysis. Using Lie group analysis we construct a class of exact solutions for this problem. We also show the previously reported modal ansatz solutions can be recovered from our present results as a special case.

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## I. INTRODUCTION

The formation of a self-written waveguide is an example of a self-action effect in which light, propagating through a photosensitive medium, is confined by the waveguide that it generates. This effect is general and requires a medium in which the refractive index increases permanently with the cumulative fluence of light. Examples of media in which self-writing has been observed include photosensitive glasses [1,2], electro-optic crystals [3] in planar geometries, photoresist [4,5], photopolymerizable resin [6], UV-cured epoxy [7], and silica glass [8] in bulk. Similar effects are also observed in photorefractive materials [9].

A self-writing effect that has been thoroughly studied is that in the planar photosensitive glass geometry. In this and related work it was found that the experimental results are consistent with a local model for the photosensitivity in which the refractive index change at some position only depends on the integrated fluence at that position. The actual relationship between these was found to be well described by [10]

$$\frac{\partial \Delta n}{\partial t} = A|\mathcal{E}|^{2p}\left(1-\frac{\Delta n}{\Delta n_s}\right), \tag{1}$$

where $\mathcal{E}$ is the electric field amplitude, $\Delta n$ is the refractive index change, $\Delta n_s$ is a saturation value for the refractive index change, and $A$ is a constant describing the strength of the photosensitivity. For one-photon processes we have $p=1$, while for two-photon processes $p=2$. The constant $A$ was found to have a small imaginary part, corresponding to a self-induced increase in the losses [10]. Both this effect and the saturation are only relevant at the late stages of waveguides' evolution; during the initial stages both effects may be neglected. To obtain a complete description of the self-writing process, we require the paraxial wave equation

$$i\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{2k}\frac{\partial^2 \mathcal{E}}{\partial y^2} + k_0\frac{\Delta n}{n_0}\mathcal{E} = 0, \tag{2}$$

which describes light propagation in the evolving planar structure. Here $z$ is the propagation direction, $y$ is the transverse direction, $k$ is the wave number, $\mathcal{E}$ is the electric field amplitude, and $\Delta n/n_0$ is the relative change in the initial refractive index $n_0$.

Apart from fully numerical approaches, insight into the solutions of the nonlinear partial differential equations (PDEs) (1) and (2) has been obtained by a Taylor series method, using $z$ as the small parameter [11], and an approach based on self-similar solutions [12], *neither of which has lead to analytical solutions*. The aim of the present work is to find such solutions. To do this, we use the method of Lie groups, one of the powerful tools available to solve nonlinear PDEs, and which can be used to derive a large class of special solutions. The exact solutions that arise from symmetry methods can often be used effectively to study properties such as asymptotics and divergences.

In this work we consider the planar geometry at the early stages of the waveguide evolution, consistent with the assumptions in an earlier work [12]. We investigate both the one-photon case [$p=1$ in Eq. (1)] with the results given in the main text, and the two-photon case [$p=2$ in Eq. (1)], with the results given in the Appendix.

To apply our analysis we first write Eq. (1) in dimensionless form. To do this we introduce $T$, $Z$, $Y$, $N$, and $E$ to be the dimensionless equivalents of the physical quantities $t$, $z$, $y$, $\Delta n$, and $\mathcal{E}$, respectively, defined by $T=a^2k_0^2n_0At(\mathcal{E}_0\mathcal{E}_0^*)^p$, $Y=y/a$, $Z=z/(k_0n_0a^2)$, $N=a^2k_0^2n_0\Delta n$, and $E=\mathcal{E}/\mathcal{E}_0$. Here $a$ is a measure of the beam width, and $\mathcal{E}_0$ is the typical large electric field amplitude. Since we consider the early evolution we may ignore $\Delta n_s$ and take $A$ to be real [see Eq. (1)]. The normalized equations are thus

$$iE_Z + \frac{1}{2}E_{YY} + NE = 0,$$

$$N_T - |E|^{2p} = 0. \tag{3}$$

To study the invariance and integrability properties of Eqs. (3), we introduce the transformation to the real functions corresponding to amplitude and phase

$$E = \mathcal{R}(Y,Z,T)e^{i\Phi(Y,Z,T)}, \tag{4}$$

so that they can be written as

$$\mathcal{R}_Z + \mathcal{R}_Y\Phi_Y + \frac{1}{2}\mathcal{R}\Phi_{YY} = 0,$$

$$\mathcal{R}_{YY} - 2\mathcal{R}\Phi_Z - \mathcal{R}\Phi_Y^2 + 2\mathcal{R}N = 0,$$

 

$$N_T - \mathcal{R}^{2p} = 0, \tag{5}$$

the set of equations used throughout the rest of the paper.

The structure of the paper is as follows. In Sec. II, we apply Painlevé analysis to the planar paraxial wave equation (5) with $p=1$. In Sec. III, we present the Lie symmetry analysis for Eqs. (5). We perform the invariance analysis for the $(1+1)$-dimensional system of PDEs and construct the resulting ordinary differential equations (ODEs) in Sec. IV. As the general ODE is nonintegrable we consider a subclass of symmetries and construct three different classes of solutions in Sec. V. In Sec. VI, we show the earlier results reported in the literature [12] can be derived from our present studies as a subcase. We present our conclusions in Sec. VII. Finally, we discuss briefly the results for $p=2$ in Eqs. (5) in the Appendix.

## II. PAINLEVÉ ANALYSIS

Here we apply Painlevé analysis to the PDEs (5) with $p=1$. Note that all references to these equations in the main text implicitly refer to this case only; in the Appendix we consider the case with $p=2$. Recall that the Painlevé test is a necessary condition for the integrability of the given nonlinear PDE [13]. As originally formulated by Ablowitz *et al.* [14], the Painlevé conjecture asserts that all similarity reductions of a completely integrable PDE should have the Painlevé property, that is, their solutions should have no movable singularities other than poles in the complex plane. A later version of the Painlevé test, the Weiss-Tabor-Carnevale (WTC) algorithm due to Weiss *et al.* [15,13], allows testing the PDEs directly, without recourse to ODEs. A PDE is said to have the Painlevé property if its solutions in the complex plane are single valued in the neighborhood of all its movable singularities.

As a first step we now effect a local Laurent expansion in the neighborhood of a noncharacteristic singular manifold $\phi(Y,Z,T)=0$, $(\phi_Y,\phi_Z,\phi_T\neq 0)$. Assuming the leading orders of the solutions of Eqs. (5) have the form

$$\mathcal{R}\sim a_0\phi^q, \quad \Phi\sim b_0\phi^r, \quad N\sim N_0\phi^p, \tag{6}$$

and substituting Eq. (6) into Eqs. (5) and balancing dominant terms, one obtains $q=-3/2$, $r=1$, $p=-2$ with the leading-order coefficients $a_0^2=15/4$, $b_0=-1$, $N_0=-15/8$.

The next step is to find the powers at which the arbitrary functions can enter into the Laurent series. Let us substitute the expression

$$\mathcal{R}=a_0\phi^{-3/2}+a_j\phi^{[j-(3/2)]}, \quad \Phi=b_0\phi+b_j\phi^{(j+1)},$$

$$N=N_0\phi^{-2}+N_j\phi^{(j-2)}, \tag{7}$$

into Eqs. (5) and considering leading-order terms alone, we obtain the set of equations

$$\begin{pmatrix} 0 & (j+1)(j-3)(a_0/2) & 0 \\ j(j-4) & 0 & 2a_0 \\ -2a_0 & 0 & (j-2) \end{pmatrix}\begin{pmatrix} \mathcal{R}_j \\ \Phi_j \\ N_j \end{pmatrix}=0. \tag{8}$$

On solving Eq. (8) one finally arrives at

$$(j+1)^2(j-3)(j^2-7j+15)=0. \tag{9}$$

The resonance $j=-1$ indicates the arbitrariness of the function in the singularity manifold $\phi$ and $j=3$ indicates that either one of the functions $a_3$, $b_3$ or $c_3$ is arbitrary in the Laurent expansion. Our analysis shows $b_3$ is arbitrary in the Laurent expansion. The remaining two resonances are complex, rather than two non-negative integers, which contradicts the assumptions of the algorithm. As a result Eqs. (5) do not admit the $P$ test and are nonintegrable.

## III. LIE SYMMETRY ANALYSIS

We investigate the invariance properties of Eqs. (5) through Lie group analysis [16–18]. Without going into the details of the theory, we present only the results below. Let us consider a one-parameter Lie group of infinitesimal transformations

$$Y\to Y' = Y+\varepsilon\xi_1(Y,Z,T,\mathcal{R},\Phi,N), \tag{10}$$

where $\varepsilon\ll 1$. Similar transformations $Z\to Z'$ and $T\to T'$ are defined, but with functions $\xi_2$ and $\xi_3$, respectively. In the same way we define the transformation

$$\mathcal{R}\to\mathcal{R}' = \mathcal{R}+\varepsilon\phi_1(Y,Z,T,\mathcal{R},\Phi,N), \tag{11}$$

again with similar transformations $\Phi\to\Phi'$ and $N\to N'$, but with functions $\phi_2$ and $\phi_3$, respectively. The invariance of Eqs. (5) under the infinitesimal point transformations (10) and (11) leads to expressions for the infinitesimals (throughout this paper we use the computer program MUMATH [19] to determine the symmetries)

$$\xi_1=c_1Y+f(Z), \quad \xi_2=2c_1Z-c_2, \quad \xi_3=-g(T),$$

$$\phi_1=-\left(c_1-\frac{1}{2}\dot{g}(T)\right)\mathcal{R}, \quad \phi_2=Yf'(Z)+h(T)+l(Z),$$

$$\phi_3=-2c_1N+Yf''(Z)+l'(Z). \tag{12}$$

Here $c_1$ and $c_2$ are arbitrary constants and $f(Z)$, $l(Z)$, $g(T)$, and $h(T)$ are arbitrary real functions of their arguments and the prime and dot denote differentiation with respect to $Z$ and $T$, respectively.

### A. Similarity variables and similarity reductions

The similarity variables associated with the infinitesimal symmetries (12) can be obtained by solving the relevant invariant surface condition or the related characteristic equation. The latter reads

$$\frac{dY}{c_1Y+f(Z)}=\frac{dZ}{2c_1Z-c_2}=\frac{dT}{-g(T)}=\frac{d\mathcal{R}}{-\left[c_1-\frac{1}{2}\dot{g}(T)\right]\mathcal{R}}$$
$$=\frac{d\Phi}{Yf'(Z)+h(T)+l(Z)}$$
$$=\frac{dN}{-2c_1N+Yf''(Z)+l'(Z)}. \quad (13)$$

We solve the characteristic Eq. (13) and construct similarity variables. Using them we can rewrite PDE (5) with originally three independent variables ($Y$, $Z$, $T$) as a system of PDE ($\tilde{\mathcal{R}}$,$\tilde{\Phi}$,$\tilde{N}$) with only two independent variables ($\eta$, $\zeta$). Solving the latter we obtain the solution for the reduced system of PDEs. From this solution we can go back to the original PDE through the similarity transformation.

Solving the characteristic Eq. (13) we find the similarity variables

$$\eta=\sqrt{\alpha(Z)}Y+F(Z),$$

$$\zeta=\int^Z\alpha(Z')dZ'+\int^TP(T')dT',$$

$$\mathcal{R}=\sqrt{\alpha(Z)}\sqrt{P(T)}\tilde{\mathcal{R}},$$

$$\Phi=\tilde{\Phi}+Q(Z)+H(T)+\eta\gamma(Z),$$

$$N=\alpha(Z)\tilde{N}+Q'(Z)+\eta\left(\gamma'(Z)-\frac{1}{2}\alpha(Z)\gamma(Z)\right)$$
$$-\frac{\alpha(Z)}{2}\gamma^2(Z), \quad (14)$$

where $\alpha(Z)=(Z-Z_1)^{-1}$ and $\gamma(Z)=-F(Z)/2-(Z-Z_1)F'(Z)$. Note that $Z_1$ is an arbitrary constant, and $P(T)$, $H(T)$, $F(Z)$, and $Q(Z)$ are four independent arbitrary real functions. These new arbitrary constants and functions can be related to the original set of arbitrary constants and functions through the relationships

$$Z_1=\frac{c_2}{2c_1},$$

$$P(T)=\frac{2c_1}{g(T)},$$

$$H(T)=-\int^T\frac{h(T')}{g(T')}dT',$$

$$F(Z)=-\frac{1}{2c_1}\int^Zf(Z')\alpha^{(3/2)}(Z')dZ',$$

$$Q(Z)=\frac{1}{2c_1}\left[\int^Z\alpha(Z')l(Z')dZ'\right.$$
$$\left.-\int^Zf'(Z')F(Z')\sqrt{\alpha(Z')}dZ'\right]. \quad (15)$$

This similarity transformation transforms Eqs. (5) to the following system of PDEs:

$$\tilde{\mathcal{R}}_\zeta+\tilde{\mathcal{R}}_\eta\tilde{\Phi}_\eta+\frac{1}{2}\tilde{\mathcal{R}}\tilde{\Phi}_{\eta\eta}-\frac{\eta}{2}\tilde{\mathcal{R}}_\eta-\frac{\tilde{\mathcal{R}}}{2}=0,$$

$$\tilde{\mathcal{R}}_{\eta\eta}+\eta\tilde{\mathcal{R}}\tilde{\Phi}_\eta-2\tilde{\mathcal{R}}\tilde{\Phi}_\zeta-\tilde{\mathcal{R}}\tilde{\Phi}_\eta^2+2\tilde{\mathcal{R}}\tilde{N}=0,$$

$$\tilde{N}_\zeta-\tilde{\mathcal{R}}^2=0. \quad (16)$$

### B. Lie vector fields and algebra

The presence of the real arbitrary functions $f(Z)$, $g(T)$, $h(T)$, and $l(Z)$ in Eq. (12) leads to an infinite-dimensional Lie algebra of symmetries. The general element of this Lie algebra can be written as

$$V=V_1+V_2+V_3(f)+V_4(g)+V_5(l)+V_6(h). \quad (17)$$

The associated vector fields are

$$V_1=\frac{\partial}{\partial Z},$$

$$V_2=Y\frac{\partial}{\partial Y}+2Z\frac{\partial}{\partial Z}-\mathcal{R}\frac{\partial}{\partial\mathcal{R}}-2N\frac{\partial}{\partial N},$$

$$V_3(f)=f(Z)\frac{\partial}{\partial Y}+Yf'(Z)\frac{\partial}{\partial\Phi}+Yf''(Z)\frac{\partial}{\partial N}, \quad (18)$$

$$V_4(g)=-g(T)\frac{\partial}{\partial T}+\frac{\mathcal{R}}{2}\dot{g}(T)\frac{\partial}{\partial R},$$

$$V_5(l)=l(Z)\frac{\partial}{\partial\Phi}+l'(Z)\frac{\partial}{\partial N},$$

$$V_6(h)=h(T)\frac{\partial}{\partial\Phi}.$$

The physical interpretation of the vectors fields (18) is the following. Vector field $V_1$ [corresponding to $c_2$ in Eq. (12)] indicates that Eqs. (5) are invariant under space translation in $Z$ whereas vector field $V_2$ (corresponding to $c_1$ in the infinitesimal symmetries) reflects the scale invariance of Eqs. (5). Since the vector field $V_3$ contains an arbitrary function $f(Z)$ its interpretation depends on $f(Z)$. For example, if $f(Z)$ is constant then $V_3$ demonstrates the space invariance of Eqs. (5) in $Y$ direction. Similarly when $g(T)$ is constant one can get time translational symmetry from $V_4$. Vector field $V_5$ demonstrates that adding an arbitrary function $l(Z)$ in phase and its derivative in refractive index $N$ does not alter the solution. The vector field $V_6$ corresponds to adding an arbitrary function $h(T)$ to the phase of $E$. The nonzero commutation relation between these vector fields are

$$[V_1,V_2]=2V_1,\quad [V_1,V_3]=V_3(f'),\quad [V_1,V_5]=V_5(l'),$$
$$[V_2,V_3]=V_3(2Zf'-f'),\quad [V_2,V_5]=2V_5(Zl'),$$
$$[V_4,V_6]=-V_6(g\dot{h}). \tag{19}$$

## IV. INVARIANCE ANALYSIS OF THE (1+1)-DIMENSIONAL EQUATION

Since Eq. (16) is a coupled system of nonlinear PDEs one may repeat the invariance analysis. In the following we investigate this equation again through Lie group analysis and explore particular solutions associated with it.

The invariance of Eq. (16) under the one-parameter Lie group of infinitesimal transformations leads to the infinitesimal symmetries

$$\xi_1=C_a,\quad \xi_2=-C_b,\quad \phi_1=0,$$
$$\phi_2=\frac{C_a}{2}\eta+C_c\zeta+C_d,\quad \phi_3=-\frac{C_a}{4}\eta+C_c, \tag{20}$$

where $\xi_i$'s and $\phi_j$'s, $i=1,\,2$, $j=1,\,2,\,3$ are the infinitesimals associated with the variables $\eta$, $\zeta$, $\tilde{\mathcal{R}}$, $\tilde{\Phi}$, and $\tilde{N}$ respectively and $C_a$, $C_b$, $C_c$, and $C_d$ are arbitrary constants.

To distinguish the vector fields associated with the symmetries (20) from (18) we represent the former by alphabetical subscripts, that is, $V_a$, $V_b$, $V_c$, $V_d$. The vector fields and nonzero commutation relation between the vector fields are

$$V_a=\frac{\partial}{\partial\eta}+\frac{\eta}{2}\frac{\partial}{\partial\tilde{\Phi}}-\frac{\eta}{4}\frac{\partial}{\partial\tilde{N}},\quad V_b=\frac{\partial}{\partial\zeta},$$
$$V_c=-\zeta\frac{\partial}{\partial\tilde{\Phi}}+\frac{\partial}{\partial\tilde{N}},\quad V_d=\frac{\partial}{\partial\tilde{\Phi}}, \tag{21}$$

and

$$[V_b,V_c]=V_d. \tag{22}$$

The vector fields $V_b$ and $V_d$ represent the $\zeta$ and $\tilde{\Phi}$ translational invariance of Eq. (16). Solving the characteristic equation associated with the symmetries (20) one obtains the similarity transformation

$$\xi=\eta+\frac{C_a}{C_b}\zeta,$$
$$\tilde{\mathcal{R}}=\hat{\mathcal{R}}(\xi),$$
$$\tilde{\Phi}=\hat{\Phi}(\xi)+\frac{C_d}{C_a}\eta+\frac{C_bC_c}{C_a^2}\xi\eta+\left(\frac{1}{4}-\frac{C_bC_c}{2C_a^2}\right)\eta^2,$$
$$\tilde{N}=\hat{N}(\xi)-\frac{\eta^2}{8}+\frac{C_c}{C_a}\eta. \tag{23}$$

Using Eq. (23) one can transform Eq. (16) to the following system of ODEs:

$$\hat{\mathcal{R}}''-\hat{\mathcal{R}}\hat{\Phi}'^2-(A+B\xi)\hat{\mathcal{R}}\hat{\Phi}'+2\hat{\mathcal{R}}\hat{N}-(C+D\xi+E\xi^2)\hat{\mathcal{R}}=0,$$
$$\hat{\mathcal{R}}\hat{\Phi}''+2\hat{\mathcal{R}}'\hat{\Phi}'+(A+B\xi)\hat{\mathcal{R}}'+I\mathcal{R}=0,$$
$$\hat{N}'-\frac{C_b}{C_a}\hat{\mathcal{R}}^2=0, \tag{24}$$

where primes denote differentiation with respect to $\xi$ and

$$A=\frac{2(C_a^2+C_bC_d)}{C_aC_b},\quad B=\frac{2C_bC_c}{C_a^2},\quad C=\frac{C_d^2}{C_a^2},$$
$$D=\frac{2C_bC_cC_d}{C_a^3},\quad E=\frac{C_b^2C_c^2}{C_a^4},$$
$$I=\frac{(4C_bC_c-C_a^2)}{2C_a^2}. \tag{25}$$

## V. PARTICULAR SOLUTIONS

In Sec. IV, we derived a similarity-reduced ODE (24) for the planar paraxial wave equation (5). Solving this equation we obtain an explicit solution for $\hat{\mathcal{R}}$, $\hat{\Phi}$, and $\hat{N}$ in terms of $\xi$, which, in turn, leads to the solution of the original problem through the transformations (23) and (14). However, as we attempt to solve the system of coupled nonlinear ODEs (24), finding its solutions depends upon whether it is integrable or not. Since $P$ analysis provides the necessary condition for integrability we apply the same to this equation. Our analysis shows that Eq. (24) does not pass the Ablowitz-Ramani-Segur algorithm [14,20] which indicates that the system (24) is nonintegrable.

We recall that our aim is to find some analytical solutions for Eqs. (5), as this helps to understand the system at least qualitatively. After a careful analysis we found that one can construct a subclass of solutions by restricting either of the parameters $C_a$ or $C_b$ to zero in the symmetries (20). As the other two parameters $C_c$ and $C_d$ do not affect the calculations we consider them nonzero in our following analysis. Below we consider the cases $C_a=0$ and $C_b=0$ separately and construct the associated solutions. Finally, we also seek a simple travelling wave reduction for Eq. (24) and show that this only leads to trivial solutions.

### A. Solution with $C_b=0$

To begin with let us consider $C_b=0$. This choice leads us to similarity variables of the form

$$\xi=\zeta,$$
$$\tilde{\mathcal{R}}=\hat{\mathcal{R}}(\xi),$$

$$\tilde{\Phi}=\hat{\Phi}(\xi)+\frac{C_d}{C_a}\eta+\frac{C_c}{C_a}\eta\zeta+\frac{\eta^2}{4},$$

$$\tilde{N}=\hat{N}(\xi)+\frac{C_c}{C_a}\eta-\frac{\eta^2}{8}. \tag{26}$$

Using the similarity transformation one can transform Eq. (16) to

$$\hat{\mathcal{R}}'-\frac{\hat{\mathcal{R}}}{4}=0,$$

$$\hat{\Phi}'-\hat{N}+\frac{1}{2}\left(\frac{C_d}{C_a}+\frac{C_c}{C_a}\xi\right)^2=0, \tag{27}$$

$$\hat{N}'-\hat{R}^2=0,$$

where prime denotes differentiation with respect to $\xi$. Equation (27) admits the general solution

$$\hat{\mathcal{R}}=I_1e^{\xi/2},$$

$$\hat{\Phi}=I_2+\left(I_3-\frac{C_d^2}{2C_a^2}\right)\xi+4I_1^2e^{\xi/2}-\frac{C_cC_d}{2C_a^2}\xi^2-\frac{C_c^2}{6C_a^2}\xi^3,$$

$$\hat{N}=I_3+2I_1^2e^{\xi/2}, \tag{28}$$

where $I_i$, $i=1,2,3$ are integration constants. Using the similarity transformations (26) and (14) in (28) one can go back to the original variables and write down the solution for the Eqs. (5). Then, also using the transformation (4), one finally arrives at

$$E=\frac{\sqrt{2c_1}I_1\exp\left[\frac{c_1}{2}\int^T\frac{dT'}{g(T')}\right]}{g^{(1/2)}(T)(Z-Z_1)^{(1/4)}}e^{i\Phi},$$

$$N=I_3\alpha(Z)+Q'(Z)+2I_1^2\alpha(Z)e^{S(Z,T)/2}+\left(\frac{C_c}{C_a}\alpha(Z)+\gamma'(Z)-\frac{1}{2}\alpha(Z)\gamma(Z)\right)R(Y,Z)-\frac{\alpha(Z)}{2}\gamma^2(Z)-\frac{\alpha(Z)}{8}R^2(Y,Z), \tag{29}$$

where

$$\Phi=I_2+Q(Z)+H(T)+\left(\frac{C_d}{C_a}+\gamma(Z)\right)R(Y,Z)+4I_1^2e^{S(Z,T)/2}+\left[\left(I_3-\frac{C_d^2}{2C_a^2}\right)-\frac{C_cC_d}{2C_a^2}S(Z,T)-\frac{C_c^2}{6C_a^2}S^2(Z,T)\right]S(Z,T)+\frac{C_c}{C_a}R(Y,Z)S(Z,T)+\frac{1}{4}R^2(Y,Z)$$

and $Q(Z)$, $H(T)$, $\gamma(Z)$, $R(Y,Z)$ $(=\eta)$, and $S(Y,Z)(=\zeta)$ are given in Eqs. (14) and (15).

Solution (29) represents a solution in which the transverse ($Y$) dependence in the refractive index $N$ does not mix with time. In other words, the refractive index evolution depends on $Z$ and $T$, but not on $Y$. This is consistent with Eq. (29) in which the only $Y$ dependence of the electric field enters through the phase $\Phi$ and, therefore, does not enter the field intensity. We return to this observation in Sec. VII.

### B. Solutions with $C_a=0$

In Sec. V A we considered the case in which the similarity variable $\xi$ depends only on the variable $\zeta$. However, one could also consider the case in which $\xi$ is a function of $\eta$ alone which can be obtained from the infinitesmial symmetries (20) by taking $C_a=0$. Repeating the calculations given in the preceding section with this restriction we obtain another class of solution of the form

$$E=\sqrt{\frac{C_c}{C_b}}\sqrt{\alpha(Z)}\sqrt{P(T)}e^{i\Phi},$$

$$N=\left(\frac{I_2^2}{2}+\frac{C_d}{C_b}\right)\alpha(Z)+Q'(Z)-\frac{\alpha(Z)}{2}\gamma^2(Z)+\frac{C_c}{C_b}\alpha(Z)S(Z,T)+\left(\frac{I_2}{2}\alpha(Z)+\gamma'(Z)-\frac{\alpha(Z)}{2}\gamma(Z)\right)R(Y,Z), \tag{30}$$

$$\Phi=I_1+Q(Z)+H(T)+[I_2+\gamma(Z)]R(Y,Z),+\frac{1}{2}R^2(Y,Z)+\left(\frac{C_d}{C_b}+\frac{C_c}{2C_b}S(Z,T)\right)S(Z,T),$$

and $Q(Z)$, $H(T)$, $R(Y,Z)$ $(=\eta)$, and $S(Z,T)(=\zeta)$ are given in Eqs. (14) and (15) and $I_1$ and $I_2$ are integration constants.

In this case also we obtain a solution in which the amplitude of $E$ is independent of $Y$ so that the refractive evolution does not depend on $Y$. A reason for obtaining this type of solution is the restriction of either one of the constants $C_a=0$ or $C_b=0$ in the infinitesimal symmetries (20). This takes us to the similarity reduced ODE that can be integrated to give $\hat{\mathcal{R}}$ as a function of $\zeta$ or constant. Rewriting this in terms of old variables we get a solution in which the amplitude part of $E$ depends only on the variables $(Z,T)$.

### C. Traveling wave reduction with $C_c$, $C_d=0$

We here consider a simple traveling wave reduction to the PDE (16) from the symmetries (20) by restricting $C_a=C_b=1$ and $C_c=C_d=0$. We now have the similarity variables that contain both $\eta$ and $\zeta$ and in particular

$$\xi=\eta+\zeta,\quad \tilde{\mathcal{R}}=\hat{\mathcal{R}},\ \ \tilde{\Phi}=\hat{\Phi}+(\eta^2/4),$$

$$\tilde{N}=\hat{N}-(\eta^2/8). \tag{31}$$

The associated similarity-reduced ODE takes the simple form

$$\hat{\mathcal{R}}'' - \hat{\mathcal{R}}\hat{\Phi}'^2 - 2\hat{\mathcal{R}}\hat{\Phi}' + 2\hat{\mathcal{R}}\hat{N} = 0,$$

$$\hat{\mathcal{R}}\hat{\Phi}'' + 2\hat{\mathcal{R}}'\hat{\Phi}' + 2\hat{\mathcal{R}}' - \frac{\hat{\mathcal{R}}}{2} = 0, \tag{32}$$

$$\hat{N}' - \hat{\mathcal{R}}^2 = 0.$$

We have applied the Ablowitz-Ramani-Segur algorithm to Eq. (32) and confirmed that it is nonintegrable. To explore special cases, we solve Eq. (32) with $\hat{\Phi}=0$. In this case one arrives at an overdetermined system of ODE, which admits a compatible solution $\hat{\mathcal{R}}=0$ and $\hat{N}=I$, where $I$ is an integration constant. As a result one obtains

$$E=0,$$

$$N = I\alpha(Z) + Q'(Z) + - \frac{\alpha(Z)}{2}\gamma^2(Z) + \left(\gamma'(Z) - \frac{1}{2}\alpha(Z)\gamma(Z)\right)R(Y,Z), \tag{33}$$

where $I$ is an integration constant and $R(Y,Z)$ $(=\eta)$ is given in Eq. (14). This class of solutions does not depend on time at all since $E=0$. In this case we therefore obtain a special case of the trivial solution $E=0$, which is consistent with any arbitrary refractive index distribution. We note that one could proceed with $\hat{N}=0$, but this also leads to a class of solutions with $E=0$.

As there are no other possibilities to consider in the infinitesimal symmetries (26) we conclude that the solutions (29) and (30) are the only nontrivial solutions one can construct for this problem through this approach.

## VI. GROUP THEORETICAL INTERPRETATION OF EARLIER RESULTS

In this section we show the results reported earlier [12] can be derived from our present studies as a special case and give group theoretical interpretation for the earlier results. In Sec. III A we constructed a general similarity transformation (14) and derived the similarity reduced $(1+1)$-dimensional PDE (16) by assuming none of the arbitrary functions or constants is zero in the infinitesimal symmetries (12). Now, restricting $f(Z)=g(T)=l(Z)=c_1=0$ and $c_2=-1$ in Eq. (12) we obtain the similarity variables

$$\eta = Y, \quad Z = T, \quad \mathcal{R} = \tilde{\mathcal{R}}(\eta,\zeta),$$

$$\Phi = \tilde{\Phi}(\eta,\zeta) + \beta(\zeta)Z, \quad N = \tilde{N}(\eta,\zeta). \tag{34}$$

Using this similarity transformation we rewrite Eqs. (5) as

$$2\tilde{\mathcal{R}}_\eta\tilde{\Phi}_\eta + \tilde{\mathcal{R}}\tilde{\Phi}_{\eta\eta} = 0,$$

$$\tilde{\mathcal{R}}_{\eta\eta} - \tilde{\mathcal{R}}\Phi_\eta^2 + 2[\tilde{N} - \beta(\zeta)]\tilde{\mathcal{R}} = 0, \tag{35}$$

$$\tilde{N}_\zeta - \tilde{\mathcal{R}}^2 = 0.$$

Restricting $\tilde{\Phi}$ to be a function of $\zeta$ alone we obtain

$$\tilde{\mathcal{R}}_{\eta\eta} + 2[N - \beta(\zeta)]\tilde{\mathcal{R}} = 0, \quad \tilde{N}_\zeta - \tilde{\mathcal{R}}^2 = 0, \tag{36}$$

which is the similarity reduction that is reported in the literature [12]. Solving Eq. (36) one obtains an explicit solution for $\tilde{\mathcal{R}}$ and $\tilde{N}$. Rewriting the solutions in terms of old variables and using the transformation (4) one can find exactly the modal ansatz discussed in the literature $E(Y,Z,T) = \mathcal{R}(Y,T)\exp[i\beta(T)Z]$ [12].

Equation (36) is invariant under the infinitesimal symmetries

$$\xi_1 = C_a\eta - C_b,$$

$$\xi_2 = -\frac{2}{\beta'(\zeta)}[C_a\beta(\zeta) + C_c],$$

$$\phi_1 = -\frac{\beta''(\zeta)\tilde{\mathcal{R}}}{[\beta'(\zeta)]^2}[C_a\beta(\zeta) + C_c], \tag{37}$$

$$\phi_2 = -2(C_a\tilde{N} + C_e),$$

where $\xi_1$, $\xi_2$, $\phi_1$, $\phi_2$ are the infinitesimals associated with the variables $\eta$, $\zeta$, $\tilde{\mathcal{R}}$, and $\tilde{N}$, respectively and $C_a$, $C_b$, $C_c$ are arbitrary constants. Solving the characteristic equation associated with the symmetries (37) leads to the similarity variables

$$\xi = (C_a\eta - C_b)\left(\beta(\zeta) + \frac{C_c}{C_a}\right)^{1/2},$$

$$\tilde{\mathcal{R}} = \hat{\mathcal{R}}(\xi)[\beta'(\zeta)]^{1/2},$$

$$\tilde{N} = \hat{N}(\xi)\left(\frac{C_c}{C_a} + \beta(\zeta)\right) - \frac{C_c}{C_a}. \tag{38}$$

Similarity transformation (38) transforms Eq. (36) to

$$\hat{\mathcal{R}}'' + \frac{2}{C_a^2}(\hat{N} - 1)\hat{R} = 0,$$

$$\xi\hat{N}' + 2\hat{N} - 2\hat{\mathcal{R}}^2 = 0, \tag{39}$$

where prime denotes differentiation with respect to $\xi$. Equation (39) with $C_a=1$ has been derived in a different manner from Eq. (36) by Monro *et al.* [12]. Here we have given a group theoretical interpretation of this result equation. We cannot construct any exact solution for Eq. (39).

## VII. DISCUSSION AND CONCLUSIONS

We carried out a detailed group theoretical analysis for the coupled PDEs (1) and (2) describing self-writing in planar structures. Even though these equations are well studied, no exact solutions were known. Through a Lie group analysis we found here a class of analytical solutions to this problem for the first time. However, these solutions either exhibit no evolution at all (Sec. V C), or describe an evolution that does not depend on the transverse coordinate $y$ (Secs. V A and V B). The latter type of solution is unlikely to be directly associated with experimental results as they would require an incident wave with infinite cross section. However, these solutions should not be dismissed for this reason as standard wave propagation can be understood using plane waves, the cross section of which is also infinite. This points to further study of the results in Secs. V A and V B: note that the presence of four arbitrary functions in the expressions leads to a very large space of solutions.

We do note that our approach here is more general than the modal ansatz of Monro *et al.* [12], discussed in Sec. VI. There the phase factor depends on $z$ and $t$, but not on $y$, whereas in our treatment the $y$ dependence is also explicitly included.

One of the main results of our analysis are the analytical solutions discussed in Sec. V. In addition to this, however, we have found the most general forms of the reduced Eqs. (16) and (24). Though we did find analytical solutions, another approach is to solve these equations numerically, subject to suitable boundary conditions at $|y|\to\infty$. Though this only allows one to find a subset of the solutions full Eqs. (1) and (2), the reduced equations are much easier to solve.


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## APPENDIX

### 1. Two photon case results

In this section, we discuss Lie symmetry analysis for the two-photon photosensitivity process, which is described by taking $p=2$ in Eqs. (5). As the procedure is analogous to the one-photon case, we report only the end results below.

#### *a. Lie symmetries, vector fields, and algebra*

The invariance of Eqs. (5) under the one-parameter Lie group of infinitesimal transformations (10) and (11) leads to the infinitesimal symmetries

$$\xi_1=\frac{f'(Z)}{2}Y+p(Z),\quad \xi_2=f(Z),\quad \xi_3=-g(T),$$

$$\phi_1=-\frac{\mathcal{R}}{4}[f'(Z)-\dot{g}(T)],$$

$$\phi_2=h(T)+l(Z)+p'(Z)Y+\frac{f''(Z)}{4}Y^2,$$

$$\phi_3=-f'(Z)N+l'(Z)+p''(Z)Y+\frac{f'''(Z)}{4}Y^2. \quad \text{(A1)}$$

In the above $f(Z)$, $p(Z)$, $l(Z)$, $g(T)$, $h(T)$, are arbitrary functions of their arguments and the prime and dot denote differentiation with respect to $Z$ and $T$, respectively. The associated vector fields are

$$V=V_1(f)+V_2(p)+V_3(g)+V_4(l)+V_5(h), \quad \text{(A2)}$$

where

$$V_1(f)=\frac{f'(Z)}{2}Y\frac{\partial}{\partial Y}+f(Z)\frac{\partial}{\partial Z}-\frac{f'(Z)}{4}\mathcal{R}\frac{\partial}{\partial \mathcal{R}}+\frac{f''(Z)}{4}Y^2\frac{\partial}{\partial\Phi}-\left(Nf'(Z)-\frac{f'''(Z)}{4}Y^2\right)\frac{\partial}{\partial N},$$

$$V_2(p)=p(Z)\frac{\partial}{\partial Y}+p'(Z)Y\frac{\partial}{\partial\Phi}+p''(Z)Y\frac{\partial}{\partial N},$$

$$V_3(g)=-g(T)\frac{\partial}{\partial T}+\frac{\mathcal{R}}{4}\dot{g}(T)\frac{\partial}{\partial R},$$

$$V_4(l)=l(Z)\frac{\partial}{\partial\Phi}+l'(Z)\frac{\partial}{\partial N},$$

$$V_5(h)=h(T)\frac{\partial}{\partial\Phi}. \quad \text{(A3)}$$

The nonzero commutation relation between the vector fields are

$$[V_1,V_2]=V_2\left(fp'-\frac{pf'}{2}\right),\quad [V_1,V_4]=V_4(fl'),$$

$$[V_3,V_5]=-gV_5(\dot{h}). \quad \text{(A4)}$$

#### *b. Similarity variables and reduction*

As discussed in Sec. III A, solving the characteristic equation associated with the symmetries (A1) one can construct the similarity transformations that now are

$$\eta=\sqrt{\alpha(Z)}Y+F(Z),$$

$$\zeta=\int^Z\alpha(Z')dZ'+\int^T P(T')dT',$$

$$\mathcal{R}=\alpha^{1/4}(Z)P^{1/4}(T)\tilde{\mathcal{R}}(\eta,\zeta), \quad \text{(A5)}$$

$$\Phi=\tilde{\Phi}+Q(Z)+H(T)+\gamma_1(Z)\eta+\gamma_2(Z)\eta^2,$$

$$N=\alpha(Z)\tilde{N}+Q'(Z)+\eta\left(\gamma_1'(Z)+\frac{\alpha'(Z)}{2\alpha(Z)}\gamma_1(Z)\right)+\eta^2[\gamma_2'(Z)-2\alpha(Z)\gamma_2^2(Z)]-\frac{\alpha(Z)}{2}\gamma_1^2(Z),$$

where $\gamma_1(Z)=-(2\gamma_2(Z)F(Z)+F'(Z)/\alpha(Z))$, $\gamma_2(Z)=-\alpha'(Z)/4\alpha^2(Z)$, and $\alpha(Z)$, $P(T)$, $H(T)$, $F(Z)$, and $Q(Z)$, are five independent arbitrary real functions. These new arbitrary functions can be related to the original set of arbitrary functions through the relationships

$$\alpha(Z)=\frac{1}{f(Z)},$$

$$P(T)=\frac{1}{g(T)},$$

$$H(T)=-\int^{T}\frac{h(T')}{g(T')}dT',$$

$$F(Z)=-\int^{Z}p(Z')\alpha^{3/2}(Z')dZ',$$

$$Q(Z)=\int^{Z}\alpha(Z')l(Z')dZ'+\frac{1}{4}\int^{Z}f''(Z')F^{2}(Z')dZ' -\int^{Z}p'(Z')\alpha^{1/2}(Z')F(Z')dZ'. \tag{A6}$$

This similarity transformation transforms Eqs. (5) to the system of PDEs,

$$\tilde{\mathcal{R}}_{\zeta}+\tilde{\mathcal{R}}_{\eta}\tilde{\Phi}_{\eta}+\frac{1}{2}\tilde{\mathcal{R}}\tilde{\Phi}_{\eta\eta}=0,$$

$$\tilde{\mathcal{R}}_{\eta\eta}-2\tilde{\mathcal{R}}\tilde{\Phi}_{\zeta}-\tilde{\mathcal{R}}\tilde{\Phi}_{\eta}^{2}+2\tilde{\mathcal{R}}\tilde{N}=0,$$

$$\tilde{N}_{\zeta}-\tilde{\mathcal{R}}^{4}=0. \tag{A7}$$

We note in the similarity reduced $(1+1)$-dimensional PDE there is no explicit presence of the independent variable unlike in the one-photon case.

#### *c. Lie symmetries and similarity reductions of Eq. (A7)*

Applying the Lie algorithm again to Eq. (A7) one obtains the Lie symmetries

$$\xi_1=C_a\eta+C_b,\quad \xi_2=2C_a\zeta-C_c,\quad \phi_1=-C_a\tilde{\mathcal{R}},$$

$$\phi_2=C_d\zeta-C_e,\quad \phi_3=-2C_a\tilde{N}+C_d, \tag{A8}$$

where $\xi_i$'s and $\phi_j$'s, $i=1,2$, $j=1,2,3$ are the infinitesimals associated with the variables $\eta$, $\zeta$, $\tilde{\mathcal{R}}$, $\tilde{\Phi}$, and $\tilde{N}$ respectively and $C_i$, $i=a,\dots,e$ are arbitrary constants.

The vector fields associated with the symmetries are

$$V_a=\eta\frac{\partial}{\partial\eta}+2\zeta\frac{\partial}{\partial\zeta}-\tilde{\mathcal{R}}\frac{\partial}{\partial\tilde{\mathcal{R}}}-2\tilde{N}\frac{\partial}{\partial\tilde{N}},\quad V_b=\frac{\partial}{\partial\eta},$$

$$V_c=\frac{\partial}{\partial\zeta},\quad V_d=\zeta\frac{\partial}{\partial\tilde{\Phi}}+\frac{\partial}{\partial\tilde{N}},\quad V_e=\frac{\partial}{\partial\tilde{\Phi}}. \tag{A9}$$

The vector field $V_a$ reflects the scale invariance of Eq. (A7) whereas vector fields $V_b$, $V_c$, and $V_e$ demonstrate the translational invariance of Eq. (A7) in the directions $\eta$, $\zeta$ and $\tilde{\Phi}$ respectively.

Solving the characteristic equation associated with the symmetries (A8) one gets the following similarity transformation:

$$\xi=\frac{(C_a\eta+C_b)}{(2C_a\zeta-C_c)^{1/2}},\quad \tilde{\mathcal{R}}=\frac{\hat{\mathcal{R}}(\xi)}{(2C_a\zeta-C_c)^{1/2}},$$

$$\tilde{\Phi}=\frac{C_d}{2C_a}\zeta+\frac{(C_cC_d-2C_a^{3}C_c)}{4C_a^{4}}\ln(2C_a\zeta-C_c)+\hat{\Phi}(\xi),$$

$$\tilde{N}=\frac{C_d\zeta}{(2C_a\zeta-C_c)}+\frac{\hat{N}(\xi)}{(2C_a\zeta-C_c)}. \tag{A10}$$

Using Eq. (A10) one can rewrite Eq. (A7) as

$$\hat{\mathcal{R}}''-\hat{\mathcal{R}}\hat{\Phi}'^{2}+\frac{2}{C_a}\xi\hat{\mathcal{R}}\hat{\Phi}'+\frac{2}{C_a^{2}}\hat{\mathcal{R}}\hat{N}+\frac{2C_e}{C_a^{2}}\hat{\mathcal{R}}=0,$$

$$\hat{\mathcal{R}}\hat{\Phi}''+2\hat{\mathcal{R}}'\hat{\Phi}'-\frac{2}{C_a}(\xi\hat{\mathcal{R}}'+\hat{\mathcal{R}})=0, \tag{A11}$$

$$\xi\hat{N}'+\frac{1}{C_a}\hat{\mathcal{R}}^{4}+2\hat{N}+\frac{C_cC_d}{C_a}=0,$$

where the prime denotes differentiation with respect to $\xi$.

#### *d. Particular solutions*

Since Eq. (A11) is nonintegrable we construct the solutions by restricting some of the parameters to zero in the infinitesimal symmetries (A8). First we take $C_a=C_c=0$. In this case we obtain

$$E=I_1\alpha^{1/4}(Z)P^{1/4}(T)e^{i\Phi},$$

$$N=I_3\alpha(Z)+Q'(Z)-\frac{\alpha(Z)}{2}\gamma_1^{2}(Z)+I_1^{4}\alpha(Z)S(Z,T) +\left(\frac{C_d}{C_b}\alpha(Z)+\gamma_1'(Z)+\frac{\alpha'(Z)}{2\alpha(Z)}\gamma_1(Z)\right)R(Y,Z)+(\gamma_2'(Z) -2\alpha(Z)\gamma_2^{2}(Z))R^{2}(Y,Z), \tag{A12}$$

where

$$\Phi=I_2+Q(Z)+H(T)+\left(\gamma_1(Z)-\frac{C_e}{C_b}\right)R(Y,Z) +\left(I_3-\frac{C_e^{2}}{2C_b^{2}}\right)S(Z,T)+\gamma_2(Z)R^{2}(Y,Z)+\left(\frac{I_1^{4}}{2}+\frac{C_dC_e}{2C_b^{2}}\right) \times S^{2}(Z,T)+\frac{C_d}{C_b}R(Y,Z)S(Z,T)-\frac{C_d^{2}}{6C_b^{2}}S^{3}(Z,T),$$

where $\alpha(Z)$, $P(T)$, $H(T)$, $F(Z)$, $Q(Z)$, $R(Y,Z)(=\eta)$, and $S(Z,T)(=\zeta)$ are defined in Eqs. (A5) and (A6).

In the second choice we restrict $C_a=C_b=0$ and obtain the following class of solution:

$$E=I_1^{1/4}\alpha^{1/4}(Z)P^{1/4}(T)e^{i\Phi},$$

$$\begin{aligned}N=&\left(\frac{I_2^2}{2}-\frac{C_e}{C_c}\right)\alpha(Z)+Q'(Z)-\frac{\alpha(Z)}{2}\gamma_1^2(Z)\\&+\left(\gamma_1'(Z)+\frac{\alpha'(Z)}{2\alpha(Z)}\gamma_1(Z)\right)R(Y,Z)\\&+[\gamma_2'(Z)-2\alpha(Z)\gamma_2^2(Z)]R^2(Y,Z)+I_1\alpha(Z)S(Z,T),\end{aligned} \tag{A13}$$

with

$$\begin{aligned}\Phi=&I_3+Q(Z)+H(T)+[I_2+\gamma_1(Z)]R(Y,Z)-\frac{C_e}{C_c}S(Z,T)\\&+\gamma_2(Z)R^2(Y,Z)+\frac{I_1}{2}S^2(Z,T),\end{aligned}$$

where $\alpha(Z)$, $P(T)$, $H(T)$, $F(Z)$, $Q(Z)$, $R(Y,Z)(=\eta)$, and $S(Z,T)(=\zeta)$ and $I_1=(C_d/C_c)$ are defined in Eqs. (A5) and (A6).

We note that the solutions (A12) and (A13) are similar to the one-photon case results. The electric field only depends on the transverse $Y$ coordinate through its phase; the evolution of the refractive index $N$ therefore does not depend on $Y$. However, the refractive index in the two-photon case solutions depend on the transverse coordinate quadratically and contain five arbitrary functions compared to one-photon case that contains only four arbitrary functions.

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